SEPARABLE G SPACES ARE ISOMORPHIC TO $C(K)$ SPACES

BY

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ABSTRACT

We show that a wide class of separable preduals of $L^1(\mu)$ spaces, namely, the G spaces, introduced by Grothendieck, are isomorphic to $C(K)$ spaces.

While the isometric theory of preduals of $L^1(\mu)$ spaces is quite developed (cf. e.g. [5, 6, 7]), very little is known about the isomorphic classification of these spaces. Only recently [1], an example of a predual of l_1 which is not isomorphic o a $C(K)$ space was constructed.

Let us recall some definitions. A closed subspace X of $C(K)$ is called a G space if there exist x_α, $y_\alpha \in K$ and numbers λ_α such that $X = \{f \in C(K) : f(x_\alpha) = \lambda_\alpha f(y_\alpha)$ for all $\alpha\}$. A subspace X of $C(K)$ is called an M space, if it is a G space with all the λ_α non-negative. A subspace X of $C(K)$ is called a $C_\sigma(K)$ space if there exists a homeomorphism σ of K onto itself with σ^2 the identity on K, such that $X = \{f \in C(K) : f(x) = -f(\sigma x)\}$.

M spaces were introduced by Kakutani [4] who proved that they coincide with the closed sublattices of $C(K)$ spaces. G spaces were introduced by Grothendieck [3].

Samuel [11] has shown that separable $C_\sigma(K)$ spaces are isomorphic to $C(S)$ spaces (see Lemma 5).

We shall prove here the following:

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*This is a part of the author's Ph.D. thesis prepared at the Hebrew University of Jerusalem under the supervision of Professor J. Lindenstrauss. I wish to thank Professor Lindenstrauss for his interest and advice.

Received November 24, 1972

THEOREM. *Every separable G space is isomorphic to a $C(K)$ space.*

We shall prove the theorem by showing that every G space is isomorphic to a $C_\sigma(K)$ space, and then use the result of Samuel mentioned above.

The question whether the theorem is true for nonseparable G spaces is left open. We do not know whether nonseparable $C_\sigma(K)$ spaces are isomorphic to $C(K)$ spaces, and whether nonseparable G spaces are isomorphic to $C_\sigma(K)$ spaces.

The proof of the theorem will be based on some lemmas.

LEMMA 1. *Let Y be a separable G space on a compact Hausdorff space H. Then there exists a compact metric space K and a G space X on K isomorphic to Y, such that:*

1) *K is the one-point compactification of the union of a sequence of mutually disjoint compact sets $\{K_n\}_{n=1}^{\infty}$.*

2) *Every $f \in X$ vanishes at infinity.*

3) *If $x \in K_n$, $y \in K_m$ and if there is a λ such that $f(x) = \lambda f(y)$ for every $f \in X$, then $|\lambda| = 2^{m-n}$, and if $n = m$ then $\lambda = -1$.*

PROOF. Let $F = \{x \in H : f(x) = 0 \ \forall f \in Y\}$. By a standard identification procedure, we can assume without loss of generality that F is at most a single point z.

Let x_α, $y_\alpha \in H$ and λ_α be the triples which define Y as a G space on H. Let f_n be a dense sequence in the unit ball of Y and put $\phi = \sum 2^{-n} |f_n|$. The function ϕ satisfies the following:

i) $\phi \geqq 0$ and $\phi(x) > 0$ whenever $x \neq z$.

ii) $\phi(x_\alpha) = |\lambda_\alpha| \phi(y_\alpha)$ for every α.

Put $H_n = \{x \in H : 2^{-n} \leqq \phi(x) \leqq 2^{-n+1}\}$, $n = 1,2,\cdots$, and let K_n be disjoint copies of the H_n's. Let K be the one point compactification of $\bigcup_{n=1}^{\infty} K_n$ with p as the point of infinity, and let $\psi : K \to H$ be the identification map, that is ψ identifies K_n with H_n and maps p into z. (Since, if the common zero z of the element of Y is an isolated point of H we can consider Y as given on $H \setminus \{z\}$, we can assume that if there is only a finite number of H_n's, then Y has no common zero. In this case, we take K to be the finite union $\cup K_n$ and proceed in exactly the same way.)

Let $\psi^0 : C(H) \to C(K)$ be the (into) isometry defined by $\psi^0 f(x) = f(\psi x)$. Denote by $C_0(K)$ the space of continuous function on K vanishing at p. The subspace $\psi^0 Y$ of $C_0(K)$ consists of all those functions f satisfying $f(u_\alpha) = \mu_\alpha f(v_\alpha)$, wherever

u_α, v_α and μ_α are such that $g(\psi u_\alpha) = \mu_\alpha g(\psi v_\alpha)$, for every $g \in Y$ (if $\psi(u_\alpha) = \psi(v_\alpha)$, we take $\mu_\alpha = 1$). Hence $\psi^0 Y$ is an isometric representation of Y as a G space on K.

Define $T: \psi^0 Y \to C_0(K)$ by

$$Tf(x) = \begin{cases} (2^n[\psi^0\phi](x))^{-1} f(x) & x \in K_n \\ 0 & x = p. \end{cases}$$

T is an isomorphism of $\psi^0 Y$ into $C_0(K)$ (since for $x \in K_n$, $1 \leqq 2^n[\psi^0\phi](x) \leqq 2$), and $T\psi^0 Y$ is again a G space. It consists of all functions satisfying $f(u_\alpha) = v_\alpha f(v_\alpha)$ where u_α, v_α are those appearing in the definition of $\psi^0 Y$ as a G space on K, and the v_α satisfy:

i) sign v_α = sign μ_α.

ii) If $u_\alpha \in K_n$, $v_\alpha \in K_m$ then $|v_\alpha| = 2^{m-n}$.

The space $X = T\psi^0 Y$ and K clearly satisfy (1) and (2) and the first claim in (3). For $x \in K$, put $[x] = \{y \in K : f(x) = f(y) \ \forall f \in X\}$. These are disjoint closed sets, and by definition, the functions in X are constant on each $[x]$. Hence we can identify each $[x]$ to a single point. Since $x \in K_n$ implies $[x] \subset K_n$, we do not identify points in different K_n's and thus (1), (2) and (3) are satisfied. Since X is separable and separates the points of K, K is metrizable.

In the sequel, we shall use the following notation: K is a compact metric space, B a closed subset of K, and τ a homeomorphism of a closed subset A of K onto itself, such that τ^2 is the identity on A and such that B is invariant under τ, that is $x \in A \cap B \Leftrightarrow \tau x \in A \cap B$.

Lemma 2. *Let K, A, B, τ be as above, and let Z be the subspace of $C(K)$ consisting of all functions f vanishing on B and satisfying $f(x) = -f(\tau x)$ for $x \in A$. Then the dual space Z^* of Z is isometric to the space of all Borel measures μ on K such that $|\mu|(B) = 0$ and such that $\mu(S) = -\mu(\tau S)$ for every Borel subset S of A. The correspondence is given by $\mu(f) = \int f d\mu$.*

Proof. Let $z^* \in Z^*$. By the Hahn-Banach theorem and the Riesz representation theorem, there exists a measure ν on K such that $|\nu|(B) = 0$ with $\|\nu\| = \|z^*\|$. Define a new measure μ by $\mu(S) = \nu(S)$ if S is a Borel subset of $K \setminus A$ and by $\frac{1}{2}(\nu(S) - \nu(\tau S))$ if S is a Borel subset of A. It is easy to see that μ is independent of the choice of ν and that the map $z^* \to \mu$ is a linear isometry of Z^* onto the set of all measures μ satisfying $|\mu|(B) = 0$ and $\mu(S) = -\mu(\tau S)$ for $S \subset A$.

LEMMA 3. *Let Z be as in Lemma 2. Then there exists a compact metric S and a homeomorphism σ of S onto itself with σ^2 the identity, such that Z is isometric to $C_\sigma(S)$.*

PROOF. For a point $k \in K$ we denote by δ_k the functional on Z defined by $\delta_k(f) = f(k)$.

Let $S = \{\pm \delta_k : k \in K \backslash B\} \cup \{0\}$ with the ω^* topology. (Notice that $\delta_k = -\delta_{\tau(k)}$ for $k \in A$, and are thus considered as a single point in S.) Clearly S is a compact metric space, and σ, defined by $\sigma(z^*) = -z^*$, is a homeomorphism of S onto itself with σ^2 the identity. The natural isometry i of Z into $C(S)$, defined by $(iz)(z^*) = z^*(z)$ is easily seen to be onto $C_\sigma(S)$.

LEMMA 4. *Let K, A, B, τ be as above, and define:*

$$Y_1 = \{f \in C(B) : f(x) = -f(\tau x) \quad \forall x \in A \cap B\}$$

$$Y_2 = \{f \in C(K) : f(x) = -f(\tau x) \quad \forall x \in A\}.$$

Then there exists a norm-preserving simultaneous extension operator T from Y_1 to Y_2 (i.e. operator $T : Y_1 \to Y_2$ with $\| T \| = 1$ such that $Tf(x) = f(x)$ for every $x \in B$).

PROOF. Let $A_1 = A \cup B$ and let $T_1 : C(B) \to C(A_1)$ and $T_2 : C(A_1) \to C(K)$ be norm-preserving simulatneous extension operators (such operators exist since K is metrizable; see [12]).

Define $T_3 : Y_1 \to C(A_1)$ by

$$T_3 f(x) = \begin{cases} f(x) & x \in B \\ \tfrac{1}{2}(T_1 f(x) - T_1 f(\tau x)) & x \in A. \end{cases}$$

Then $\| T_3 \| = 1$ and if $f \in Y_1$, $T_3 f$ is an element of $C(A_1)$ satisfying $T_3 f(x) = - T_3 f(\tau x)$ for every $x \in A$. The operator $T = T_2 \, T_3$ has the desired properties.

The following result is due to Samuel [11]; for the sake of completeness, we supply a proof (which is essentially due to Fakhoury [2]).

LEMMA 5. *Every separable $C_\sigma(K)$ space is isomorphic to a $C(H)$ space.*

PROOF. Let $X = C_\sigma(K)$ with K compact metric. There is no loss of generality to assume that X separates the points of K. We consider two cases:

Case I. K is uncountable.

We can find an uncountable closed set S such that $\sigma S \cap S = \varnothing$. Let T be the extension operator we get from Lemma 4 by taking $A = K$, $\tau = \sigma$ and $B = S \cup \sigma S$, and let $T_1: C(S) \to C(S \cup \sigma S)$ be defined by

$$T_1 f(x) = \begin{cases} f(x) & x \in S \\ -f(\sigma x) & x \in \sigma S. \end{cases}$$

Then TT_1 is an isometric embedding of $C(S)$ into $C_\sigma(K)$.

By a theorem of Milutin [9], $C(S)$ is isomorphic to $C(K)$, and since $C_\sigma(K)$ is complemented in $C(K)$ (with the projection $Pf(x) = \frac{1}{2}(f(x) - f(\sigma x))$, we get by a theorem of Pełczyński [10] that $C_\sigma(K)$ is isomorphic to $C(K)$.

Case II. K is countable.

By [8], there exists a countable ordinal α such that K is homeomorphic to the set $[\alpha]$ of all ordinals ξ with $\xi \leqq \alpha$, equipped with the order topology. Let $\beta_0 \leqq \alpha$ be the fixed point if σ (if there is one), and define

$$K_1 = \{\beta \leqq \alpha : \beta \leqq \sigma(\beta)\}$$
$$K_2 = \{\beta \leqq \alpha : \beta \geqq \sigma(\beta)\}.$$

Then K_1, K_2 are closed, $[\alpha] = K_1 \cup K_2$, $K_1 \cap K_2 = \{\beta_0\}$ and σ is a homeomorphism of K_1 onto K_2.

The operator T defined by

$$Tf(x) = \begin{cases} f(x) & x \in K_1 \\ -f(\sigma x) & x \in K_2 \end{cases}$$

is an isometry of $C_0(K_1)$ (the space of all continuous functions on K_1 vanishing at β_0) onto $C_\sigma(K)$. Since $C_0(K_1)$ is isomorphic to $C(K_1)$, we get that $C_\sigma(K)$ is isomorphic to $C(K_1)$.

Proof of the Theorem. Let X be a separable G space, and assume that it is given on a space K as in Lemma 1. Define closed subsets A_n and B_n of K_n by

$$A_n = \{x \in K_n : \exists y \in K_n \text{ such that } f(x) = -f(y) \ \ \forall f \in X\} \ \ n = 1,2,\cdots,$$
$$B_n = \{x \in K_n : \exists \lambda \ \exists y \in \bigcup_{m<n} K_m \text{ such that } f(x) = \lambda f(y) \ \ \forall f \in X\} \ \ n = 2,3\cdots.$$

Put $A = \overline{\cup A_n}$, $B = \overline{\cup B_n}$ and for $x \in A$, define τx to be the unique point satisfying $(x) = -f(\tau x)$ for every $f \in X$ (define $\tau p = p$ where p is the point of infinity). It is easy to check that K, A, B, τ satisfy the assumptions of Lemmas 2–4 and that for each n, τ maps A_n onto itself.

Let

$$Y_1 = \{f \in C(B) : f(x) = -f(\tau x) \ \ \forall x \in A \cap B\}$$
$$Y_2 = \{f \in C(K) : f(x) = -f(\tau x) \ \ \forall x \in A\}$$
$$Z \ = \{f \in Y_2 : f \text{ vanishes on } B\}.$$

Let T: $Y_1 \to Y_2$ be the extension operator given by Lemma 4 and R: $C(K) \to C(B)$ be the restriction operator.

Define $S: X \to Z$ by $S = I - TR$. Since $\| T \| = 1$ and since $\| Rf \| \leqq \frac{1}{2} \| f \|$ for $f \in X$, we get that $\frac{1}{2} \| f \| \leqq \| Sf \| \leqq 3/2 \| f \|$ for every $f \in X$, and hence, S is an isomorphism of X into Z. By Lemmas 3 and 5, we only have to show that S is onto Z.

If $SX \neq Z$, then there exists a $\mu \in Z^*$, $\| \mu \| = 1$, such that μ annihilates SX. Since by Lemma 2 $|\mu|(B) = 0$, we can find closed subsets C_n of K_n with $C_n \cap B_n = \emptyset$ such that $|\mu|(\cup C_n) < 1 - 1/10$, and since every B_n is invariant under τ, we can also assume that C_n is invariant under τ.

Let f be a continuous function on $C = \overline{\cup C_n}$, vanishing at p with $\| f \| = 1$, and such that $\int f d\mu > 1 - 2/10$. Since μ is antisymmetric on subsets of A, we can assume that $f(x) = -f(\tau x)$ whenever $x \in A \cap C$. We shall show that f can be extended to an element f of X with $\| f \| = 1$. We define f on K_n by induction on n. For $n = 1$, we use Lemma 4 to extend f from C_1 to K_1 preserving its norm and the relation $f(x) = -f(\tau x)$ for $x \in A_1$. After n steps, f is already defined on $\cup_{m \leqq n} K_m$ and has a unique extension to B_{n+1} preserving the relations satisfied by the elements of X. Again we use Lemma 4 to extend f from $C_{n+1} \cup B_{n+1}$ to all K_{n+1}.

In this way, we get a function f defined on K, continuous on each K_n and satisfying all the relations satisfied by elements of X, and we only have to show that $\sup \{|f(x)| : x \in K_n\}$ tends to zero.

Let N be such that $\sup \{|f(x)| : x \in C_n\} \leqq \frac{1}{2}$ for $n > N$. Sinse also $|f(x)| \leqq \frac{1}{2}$ for each $x \in B_n$, and since the extension was norm preserving on each K_n, it follows that $\sup \{|f(x)| : x \in K_n\} \leqq \frac{1}{2}$ for every $n > N$. A similar argument shows that if N is such that $n > N$ implies $\sup \{|f(x)| : x \in C_n\} \leqq 2^{-k}$, then $\sup \{|f(x)| : x \in K_n\} \leqq 2^{-k}$ for each $n \geqq N + k$, which proves that f is also continuous at p and thus belongs to X.

Now $\mu(f) \geqq \int_C f d\mu - \int_{K \setminus C} |f| d|\mu| \geqq 1 - 3/10$ and since $\| TRf \| \leqq \frac{1}{2}$, we get that $|\mu(TRf)| \leqq \frac{1}{2}$. Hence we get that $\mu(Sf) = \mu(f) - \mu(TRf) \geqq 1 - 3/10 - 1/2 > 0$, a contradiction. Q.E.D.

Note Added in Proof: It should be noted that the proof given here applies to real valued functions only. However, by suitable modifications, it can be shown that the theorem is true in the complex case as well.

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